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Quantum Chaos Versus Noise Environment

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Quantum dynamics of a classically chaotic 1D system in the presence of external noise is studied. Stability and reversibility properties of the motion (characterized by the Peres fidelity) as functions of the noise level σ are considered. We calculate fidelity analytically in the cases of weak and very strong noise and find critical value, $\sigma_c(t)$, below which the effect of perturbation remains small. Decay of critical perturbation with time is found to be power-like after the Ehrenfest time t_E. An estimation of the decoherence time $t_d(\sigma)$ is presented after which the averaged density matrix becomes diagonal and its evolution turns into a Markovian process.

Keywords: quantum chaos, Wigner function, Fidelity, stability, reversibility, markov's chain, entropy, purity.

Introduction

Classical dynamical chaos caused by the exponential sensitivity of the motion to arbitrary weak perturbations results in practical *irreversibility* [1]. On the contrary, the quantum dynamics is discovered to be much more stable with respect to perturbations and therefore be fairly *reversible* [2].

Recent observation [3, 4] showed that the quantum evolution is quite stable and time-reversible as long as the strength ξ of an instant perturbation applied at the moment t does not exceed a *critical* value $\xi_c(t) = \dfrac{\sqrt{2}}{\mathcal{M}(t)}$. Here the number $\mathcal{M}(t)$ of angular harmonics of the quantum Wigner function characterizes complexity of the quantum state at the moment of time t. Therefore there exists direct relation between sensitivity of dynamics to perturbations and complexity of the quantum state.

The number of harmonics $\mathcal{M}(t)$ grows exponentially with a rate that corresponds to the classical Lyapunov exponent up to the Ehrenfest time t_E only. After the Ehrenfest time the growth becomes merely a power-like one, and the corresponding critical value $\xi_c(t)$ of the perturbation decreases with time rather slow that implies sufficiently good stability of quantum dynamics.

Typically, the perturbation is not a single but instead of this it is a stationary and random one (a noise). The goal of this study is to explore stability and reversibility of chaotic quantum dynamics with respect to a persistent external noise.

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1. Model

1.1. Nonlinear Quantum Oscillator Driven by Periodic Kicks

As an illustrative example we consider a nonlinear oscillator driven by periodic kicks. The Hamiltonian of the oscillator (in the absence of noise) reads

$$\hat{H}(\hat{a},\hat{a}^\dagger,t) = \hbar\omega_0\hat{n} + \hbar^2\hat{n}^2 - \sqrt{\hbar}g(t)(\hat{a}+\hat{a}^\dagger) = \hat{H}_0 + \hat{H}_{kick}, \tag{1}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}^\dagger$ and $\hat{a}^\dagger$, $\hat{a}^\dagger$ are bosonic creation-annihilation operators. The driving force $g(t) = g_0\sum_s \delta(t-s)$ is a periodic sequence of instant kicks. Floquet one-step evolution operator is

$$\hat{\mathcal{F}} = e^{-i\left(\omega_0\hat{n}+\hbar\hat{n}^2\right)} e^{i\frac{g_0}{\sqrt{\hbar}}(\hat{a}+\hat{a}^\dagger)} = e^{-\frac{i}{\hbar}\hat{H}_0}\hat{D}\left(\frac{ig_0}{\sqrt{\hbar}}\right), \tag{2}$$

so that the density matrix evolves as $\hat{\rho}(t) = \hat{\mathcal{F}}^t\hat{\rho}(0)\hat{\mathcal{F}}^{\dagger t}$. We suppose below the initial state to be the ground state $\hat{\rho}(0) = |0\rangle\langle 0|$ of the operator $\hat{H}_0$.

The classical dynamics of such an oscillator becomes *chaotic* when the strength g_0 exceeds a critical value $g_{0,c} \approx 1$. The decay of angular phase correlations in this case is exponential and the mean action grows diffusively with the diffusion coefficient $D = g_0^2$. In this paper we consider quantum dynamics in the case when the classical chaos is well-developed.

Numerical observations show that the *coarse-grained* distribution of the excitation numbers n is practically exponential [3]:

$$w_n(t) \equiv \rho_{nn}(t) \approx \frac{1}{\langle n\rangle_t + 1}\left[\frac{\langle n\rangle_t}{\langle n\rangle_t + 1}\right]^n, \tag{3}$$

where the mean excitation number is $\langle n\rangle_t = \sum\limits_{n=1}^{\infty} n\rho_{nn}(t)$.

As in [3] we characterize the *complexity* of the quantum state by the mean number $\mathcal{M}(t) = \sqrt{\langle m^2\rangle_t} = \sqrt{\sum\limits_{m=1}^{\infty} m^2\mathcal{W}_m(t)}$ of angular harmonics of the Wigner function. The corresponding distribution $\mathcal{W}_m(t)$ is given by

$$\mathcal{W}_{m\geqslant 0}(t) \equiv (2-\delta_{m0})\frac{\sum\limits_{n=0}^{\infty}|\langle n+m|\hat{\rho}(t)|n\rangle|^2}{\sum\limits_{n=0}^{\infty}|\langle n|\hat{\rho}^2(t)|n\rangle|}. \tag{4}$$

1.2. Noise

We introduce the noise $\hat{V}_{noise} = \xi_t\hbar\hat{n}\sum_s\delta(t-s)$ as a sequence of instant perturbations with random amplitudes. At a moment t such a perturbation generates rotation in the phase space $\hat{P}(\xi_t) = e^{-i\xi_t\hat{n}}$ by a random angle ξ_t. The angles ξ_t are supposed to be uncorrelated $\overline{\xi_t\xi_{t'}} = \sigma^2\delta_{tt'}$ normally distributed random variables $p(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\xi^2}{2\sigma^2}}$ (white noise). The time evolution operator in the presence of the noise

$$\hat{U}(t;\xi) = \prod_{\tau=1}^{t}\hat{\mathcal{F}}(\xi_\tau) = \prod_{\tau=1}^{t}\left[e^{-i\xi_\tau\hat{n}}\hat{\mathcal{F}}\right] \tag{5}$$

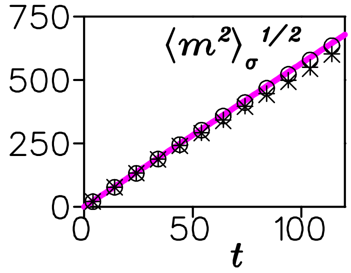

Fig. 1. Time evolution of the number of harmonics $\sqrt{\langle m^2\rangle_\sigma} \approx \sqrt{2}\langle n\rangle_{t;\sigma}$ in the cases with no noise (crosses, $\sigma = 0$), weak (pluses, $\sigma = 0.001$) and strong noise (circles, $\sigma = \infty$) as compared with the classical diffusion law (straight magenta lines)

is unitary for any fixed noise history $\xi \equiv \{\xi_1, \xi_2, \ldots, \xi_t\}$ so that the state $\hat\rho(t;\xi) = \hat U(t;\xi)\hat\rho(0)\hat U^\dagger(t;\xi)$ remains pure during the whole evolution.

Since we study the influence of the noise on the system dynamics it is interesting to trace how different characteristics of the motion depend on the noise history. We see from the numerical data (Fig. 1) that excitation number $\langle n\rangle_{t;\xi}$ as well as the mean number of harmonics $\sqrt{\langle m^2\rangle_{t;\xi}} \approx 2\langle n\rangle^2_{t;\xi}$ do not depend on the realization ξ and therefore they are self-averaged quantities. Moreover, these quantities depend on the noise level σ rather weakly, i.e. $\langle n\rangle_{t;\sigma=0} \approx \langle n\rangle_{t;\sigma=\infty}$ and $\sqrt{\langle m^2\rangle_{t;\sigma=0}} \approx \sqrt{\langle m^2\rangle_{t;\sigma=\infty}}$.

2. Stability

Stability of the motion with respect to the noise could be characterized by the overlap $F(t;\xi) = \mathrm{Tr}[\hat\rho(t)\hat\rho_\xi(t)]$ of the states developed by the moment t during the evolutions with and without noisy perturbation. However, contrary to the quantities $\langle n\rangle_{t;\xi}$ and $\sqrt{\langle m^2\rangle_{t;\xi}}$, the fidelity $F(t;\xi)$ strongly depends on the noise realization and therefore is not self-averaging one. Thus the noise-averaged fidelity is appropriate measure of dynamics' stability

$$F(t;\sigma) = \overline{\mathrm{Tr}[\hat\rho(t)\hat\rho_\xi(t)]} = \mathrm{Tr}[\hat\rho(t)\hat\rho^{(av)}(t;\sigma)]. \tag{6}$$

This definition implicates averaged density matrix $\hat\rho^{(av)}(t;\sigma) = \overline{\hat\rho(t;\xi)}$.

It is easy to see that one-step evolution of the averaged density matrix is given by the following recurrence relation

$$\begin{aligned}\rho^{(av)}_{nn'}(\tau;\sigma) &= \overline{\langle n|e^{-i\xi_\tau\hat n}\hat{\mathcal F}\hat\rho^{(av)}(\tau-1;\sigma)\hat{\mathcal F}^\dagger e^{i\xi_\tau\hat n}|n'\rangle} = \\ &= e^{-\frac{\sigma^2}{2}(n-n')^2}\langle n|\hat{\mathcal F}\hat\rho^{(av)}(\tau-1;\sigma)\hat{\mathcal F}^\dagger|n'\rangle\end{aligned} \tag{7}$$

that describes non-unitary dynamics of $\hat\rho^{(av)}(t;\sigma)$. The normalization condition $\mathrm{Tr}[\hat\rho^{(av)}(\tau;\sigma)] = 1$ holds here for every τ and σ.

Numerical simulations prove that the *coarse-grained* distribution $w_n^{(av)}(t;\sigma)$ (similar to the case (3) of the dynamics without noise) is exponential:

$$w_n^{(av)}(t;\sigma) \approx \frac{1}{\langle n\rangle_{t;\sigma}+1}\left[\frac{\langle n\rangle_{t;\sigma}}{\langle n\rangle_{t;\sigma}+1}\right]^n \tag{8}$$

where $\langle n\rangle_{t;\sigma} = \sum\limits_{n=1}^{\infty} n w_n^{(av)}(t;\sigma) = \overline{\langle n\rangle_{t;\xi}}$ is the noise-averaged excitation number.

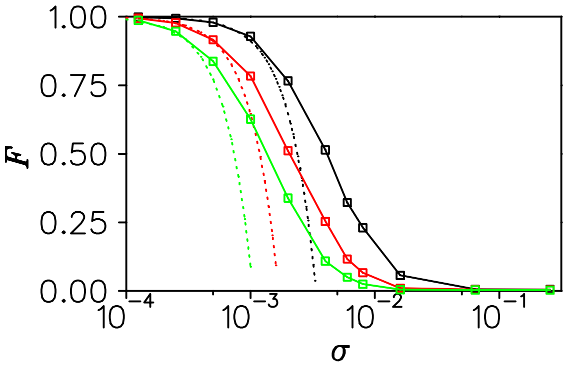

Fig. 2. Fidelity $F(t;\sigma)$ decay vs noise strength σ for different moments of time (log-linear plot). Black, red and green solid lines correspond to $t = 10, 35, 60$. Weak noise approximation (9) is also presented (dotted lines)

Analytical expressions for the fidelity $F(t;\sigma)$ can be obtained in the two limiting cases of very weak and very strong noise.

2.1. Weak Noise

While calculating fidelity in the lowest order with respect to the level σ of the noise we keep perturbation ξ_τ only in one exponential factor in the (5) thus reducing the problem to that considered in [3]. For a single perturbation with a small strength ξ_τ applied at the instant τ fidelity reads $F(t;\xi_\tau) = 1 - \frac{\xi_\tau^2}{2}\langle m^2\rangle_\tau + \mathcal{O}(\xi_\tau^4)$ [3]. Summing then over all moments $1 \leqslant \tau \leq t$ and averaging over ξ_τ we immediately obtain

$$F(t;\sigma) = 1 - \frac{1}{2}\sigma^2\sum_{\tau=1}^{t}\langle m^2\rangle_\tau + \mathcal{O}(\sigma^4) = 1 - \frac{\sigma^2}{\sigma_c^2(t)} + \ldots, \tag{9}$$

where the *critical* perturbation strength is defined as $\sigma_c(t) = \sqrt{2 \,/\, \sum_{\tau=0}^{t}\langle m^2\rangle_\tau}$ and the mean values $\langle m^2\rangle_\tau$ are taken in the absence of the noise. While deviation of fidelity from unity is small, this estimate agrees well with data, see Fig.2. After the Ehrenfest time the relation $\langle m^2\rangle_\tau \approx 2\langle n\rangle_\tau(\langle n\rangle_\tau + 1)$ is valid [3] between the number of harmonics and degree of excitation with the same accuracy as exponential anzatz (3). Since, as has been mentioned above, the mean excitation number does not practically depend on the noise level we arrive at $\langle n\rangle_\tau^2 \approx \langle n\rangle_{\tau;\infty}^2 = \left(\frac{g_0^2}{\hbar}\right)^2\tau^2$. As a result $\langle m_\tau^2\rangle \approx 2\left(\frac{g_0^2}{\hbar}\right)^2\tau^2$ and $\sigma_c(t) \approx \left[\frac{\sqrt{3}\hbar}{g_0^2}\right]t^{-3/2} \propto t^{-3/2}$, i.e. the critical perturbation decreases with time power-like when $t > t_E$.

2.2. Strong Noise (Markovian Limit)

In the recurrence relation (7) the factor $e^{-\frac{\sigma^2}{2}(n-n')^2} \xrightarrow{\sigma\to\infty} \delta_{nn'}$ and the density matrix remains diagonal $\hat{\rho}^{(av)} \Rightarrow \hat{\rho}^{(d)}$ during the whole evolution. One-step evolution equation reads

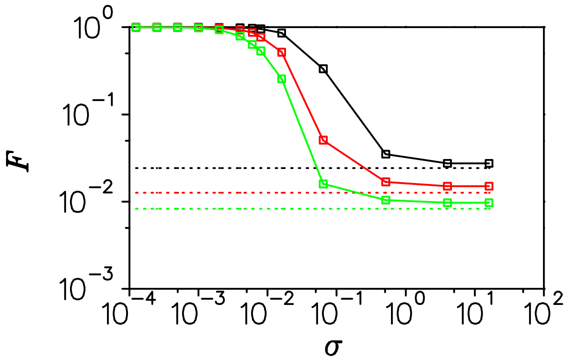

Fig. 3. Fidelity $F(t;\sigma)$ vs σ for the different moments of time (log-log plot). Black, red and green solid lines correspond to $t = 5, 10, 15$. Asymptotics (11) for the strong noise are presented (dotted lines)

now

$$w_n^{(d)}(\tau) = \sum_{k=0}^{\infty} |\langle n|\hat{D}\left(\frac{ig_0}{\sqrt{\hbar}}\right)|k\rangle|^2 w_k^{(d)}(\tau-1). \tag{10}$$

This expression contains only transition probabilities in the absence of noise. The extremely strong noise destroys coherence of the quantum state and the density matrix evolution turns into Markovian process. The dynamics is fully determined by the kick operator and nonlinearity of the system is inessential in this limit.

The fidelity (6) can be easily calculated with the help of the exponential anzatz (8):

$$F_\infty(t) = \mathrm{Tr}[\hat{\rho}_0(t)\hat{\rho}^{(d)}(t)] = \frac{1}{1 + \langle n\rangle_{t;0} + \langle n\rangle_{t;\infty}}, \tag{11}$$

where $\langle n\rangle_{t;0}$ is the mean excitation number in the absence of noise and $\langle n\rangle_{t;\infty} = \frac{g_0^2}{\hbar}t$ corresponds to the classical-like diffusion induced by the noise.

Such an asymptotical behavior corresponds to full stirring over all the states available at the moment t.

2.3. Moderate Noise

It is impossible to calculate $F(t;\sigma)$ analytically for intermediate values of σ. Numerically, we observe some *scaling law*: the fidelity $F(t;\sigma)$ depends on the only variable $[\sigma/\sigma_c(t)]$ in a wide domain of σ up to a new critical value $\tilde{\sigma}_c(t)$, where the fidelity decay changes to the asymptotic law $F_\infty(t)$ (11) (see Fig.3). A simple fit

$$F[\sigma/\sigma_c(t)] = \frac{1}{1 + \sigma^2/\sigma_c^2(t)} \tag{12}$$

describes our numerical data rather well (see Fig. 4). The second critical value defines the decoherence time $t_d(\sigma) = \frac{\sqrt{6}\hbar}{g_0\sigma}$ when the transition $F[\sigma/\sigma_c(t_d)] = F_\infty(t_d)$ from the scaling regime to strong noise behavior takes place.

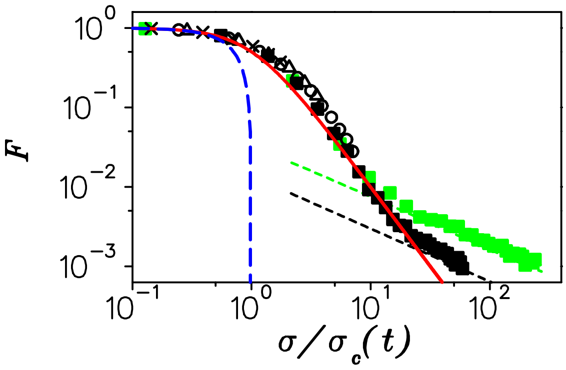

Fig. 4. Fidelity data vs scaling factor $\sigma/\sigma_c(t)$. Crosses, triangles, open circles and squares correspond to the different levels of the noise: $\sigma = (0.5, 1, 2, 16, 64) \times 10^{-3}$. The blue dashed line is the weak noise approximation (9), the black and green dashed lines are the strong noise asymptotics (11) for the two latter values of σ. The red line is the fit (12)

3. Reversibility and Purity

Reversibility of quantum dynamics is characterized by the overlap of the initial $\hat{\rho}(0)$ and reversed $\hat{\rho}(t|0;\xi,\xi') = \hat{U}^{\dagger}_{\xi'}(t)\hat{\rho}(t;\xi)\hat{U}_{\xi'}(t)$ at the moment t states: $F^{(rev)}(t;\sigma) = \overline{\mathrm{Tr}[\hat{\rho}(0)\hat{\rho}(t|0;\xi,\xi')]}$. As the noise histories ξ and ξ' are uncorrelated we have

$$\begin{aligned} F^{(rev)}(\sigma;t) &= \overline{\mathrm{Tr}\left[\hat{\rho}(0)\hat{\rho}(0|t;\xi,\xi')\right]} = \\ &= \overline{\mathrm{Tr}\left[\hat{\rho}(t;\xi)\hat{\rho}(t;\xi')\right]} = \\ &= \mathrm{Tr}\left[\hat{\rho}^{(av)}(t;\sigma)\hat{\rho}^{(av)}(t;\sigma)\right] \equiv \mathbb{P}(\sigma;t). \end{aligned} \tag{13}$$

The quantity $\mathbb{P}(t;\sigma)$ is referred for as purity [5]. Using as usual the exponential anzatz we get the purity as a function of σ:

$$\mathbb{P}(t;\sigma) = \frac{2\langle|\mathfrak{m}|\rangle_{t;\sigma} + 1}{2\langle n\rangle_{t;\sigma} + 1}, \tag{14}$$

where the quantity $\langle|\mathfrak{m}|\rangle_{t;\sigma} = \sum_{m=1}^{\infty} m\,\mathbb{W}_m(t;\sigma)$ is the mean number of harmonics of the *averaged* Wigner function. The probability distribution $\mathbb{W}_m$ is defined by the averaged density matrix

$$\mathbb{W}_m(t;\sigma) \equiv (2-\delta_{m0})\frac{\sum_{n=0}^{\infty} |\langle n+m|\hat{\rho}^{(av)}(t;\sigma)|n\rangle|^2}{\sum_{n=0}^{\infty} |\langle n|[\hat{\rho}^{(av)}(t;\sigma)]^2|n\rangle|}. \tag{15}$$

In the strong noise limit the averaged density matrix becomes diagonal, so the only zero harmonic survives $\mathbb{W}_m(t;\infty) = \delta_{m0}$ and

$$\mathbb{P}(t;\sigma) = F^{(rev)}(t;\sigma) = \mathrm{Tr}[\hat{\rho}^{(d)}(t)]^2 = \frac{1}{1+2\langle n\rangle_{t;\infty}}. \tag{16}$$

4. Entropy

Instead of $\mathcal{M}(t)$, complexity of the quantum state can also be described in the absence of noise in terms of the information Shannon entropy defined as $\mathcal{I}(t) = -\sum_{m=0}^{\infty} \mathcal{W}_m(t)\ln\mathcal{W}_m(t)$.

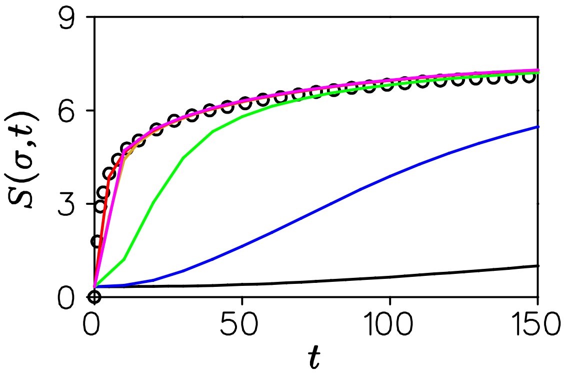

Fig. 5. Von Neumann entropy $S(t)$ for the different noise levels $\sigma = [0.125, 1, 8, 64, 512, 4000] \times 10^{-3}$. Limiting curve (circles) corresponds to the Shannon entropy $\mathcal{I}(t)$ without noise

Using the exponential anzatz (3) and relation $\mathcal{W}_m(t) = (2-\delta_{mo})\sum\limits_{m=0}^{\infty}\rho_n(t)\rho_{n+m}(t)$ we find that $\mathcal{W}_m \approx \dfrac{(2-\delta_{mo})}{2\langle n\rangle_{t;0}+1}\left[\dfrac{\langle n\rangle_{t;0}}{\langle n\rangle_{t;0}+1}\right]^n$ and finally $\mathcal{I}(t) \approx \ln\langle n\rangle_{t;0} + 1 + \frac{1}{2\langle n\rangle_{t;0}} + ...\ (t \gg 1)$.

From the other hand reversibility of dynamics is closely related to the von Neumann invariant entropy $S(t;\sigma) = -\mathrm{Tr}[\hat{\rho}^{(av)}\ln\hat{\rho}^{(av)}]$. This entropy monotonically grows with time (see Fig. 5) and approaches from below the Shannon entropy when $t \to t_d(\sigma)$. After the decoherence time the evolution becomes Markovian and

$$S_\infty(t) \approx -\sum_{n=0}^{\infty} w_n^{(d)}(t)\ln w_n^{(d)}(t) \approx \ln\langle n\rangle_{t;\infty} + 1 + \frac{1}{2\langle n\rangle_{t;\infty}} + ... = \mathcal{I}(t). \tag{17}$$

In particular, in the strong noise limit $\sigma \to \infty$ the decoherence time $t_d(\infty) = 0$ so that $S_\infty(t) = \mathcal{I}(t)$ for any time $0 < t < \infty$.

Similar correspondence between information and correlational entropies has been found in the theory of random band matrices [6].

Conclusion

Quantum dynamics of a classically chaotic system after the Ehrenfest time has been studied in the presence of noise. We have shown numerically that such global characteristics of evolution as the degree of excitation $\langle n\rangle_{t;\xi}$ and the mean number of harmonics $\sqrt{\langle m^2\rangle_{t;\xi}}$ do not depend on the noise history, so they are self-averaging quantities. We have proved that the coarse-grained distribution $w_n^{(av)}(t;\sigma) \equiv \rho_{nn}^{(av)}(t;\sigma)$ obeys the universal exponential law. Stability and reversibility properties of dynamics have been investigated using the noise-averaged Peres fidelity. Analytical expressions have been obtained in two limiting cases of weak and strong noise. We have found a critical perturbation $\sigma_c(t) = \sqrt{2 \,/\, \sum\limits_{\tau=0}^{t}\langle m^2\rangle_\tau}$ below which the effect of the noise remains small. This critical perturbation decays with time in accordance with power law $\sigma_c(t) \propto t^{-3/2}$ contrary to the exponential decay in the classical limit. However the decay caused by noise is faster than in the case of a single perturbation when $\xi_c(t) \propto t^{-1}$. In the limit $\sigma \to \infty$ the density matrix $\rho_{nn'}^{(av)}(t;\sigma)$ becomes diagonal and its evolution turns into Markovian process. For the intermediate values of the noise level we have discovered a scaling law: the fidelity $F(t;\sigma)$ depends on the only variable $[\sigma/\sigma_c(t)]$ in a wide domain of σ up to some new critical value $\tilde{\sigma}_c(t)$ where transition to the Markovian regime takes place. This allowed us to estimate the time $t_d(\sigma)$ of full decoherence.

The relation between degree of reversibility and purity of the quantum state, $F^{(rev)}(t;\sigma) = \mathbb{P}(t;\sigma)$, has been established. We have also proved that the von Neumann entropy coincides in the limit of the strong noise with the information Shannon entropy $\mathcal{I}(t) = S_\infty(t)$.

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Квантовый хаос при наличии внешнего шума

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Олег В. Жиров

В работе изучена квантовая динамика классически хаотичной системы при наличии внешнего шума. Рассматриваются свойства устойчивости и обратимости движения (характеризуемые величиной фиделити Переса) в зависимости от уровня шума σ. Мы получили аналитическое выражение для фиделити в случаях слабого и сильного шума, а также нашли критическое значение, $\sigma_c(t)$, ниже которого влияние возмущения остаётся малым. Было обнаружено, что после времени Эренфеста t_E критическое возмущение падает со временем по степенному закону. Представлена оценка для времени декогерентности $t_d(\sigma)$, после которого матрица плотности становится диагональной, а ее эволюция является марковским процессом.

Ключевые слова: квантовый хаос, функция Вигнера, фиделити, стабильность, обратимость, марковская цепь, чистота состояния.